Note on Bessel functions of type A_{N-1}.

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Abstract

Through the theory of Jack polynomials we give an iterative method for integral formula of Bessel function of type A_{N-1} and a partial product formula for it. [1]

1 Introduction and backgrounds

Dunkl operators which were first introduced by C. F. Dunkl [6] in the late 80ies are commuting differential-difference operators, associated to a finite reflection groups on a Euclidean space. Their eigenfunctions are called Dunkl kernels and appear as a generalization of the exponential functions. Although attempts were made to study them and except the reflection group $\mathbb{Z}_2^N$ the explicit forms or behaviors of these kernels are remain unknown. In the present work we will be concerned with generalized Bessel functions J_k defined through symmetrization of Dunkl kernels in the case of the symmetric group S_N. We will obtain the following

$$J_k(\mu,\lambda) = \int_{\mathbb{R}^{N-1}} e^{\langle \mu, x\rangle} \delta_k(\lambda, x) dx. \tag{1.1}$$

where the function δ_k can be explicitly computed using a recursive formula on the dimension N. The key ingredient is the integral formula of A. Okounkov and G. Olshanski [10] for Jack polynomials. As the last are connected with Heckman-opdam-Jacobi polynomials [2] the formula (1.1) follow by limit transition. We should note here that when $N = 3$, the formula (1.1) is comparable to that obtained by C. F. Dunkl [5] for intertwining operator.

Let us start with some well-known facts about Heckman Opdam Jacobi polynomials, Jack polynomials and Dunkl kernels associated with a root system R. The standard references are [2, 4, 8, 11, 16, 15]. Here $\mathbb{R}^N$ is equipped with the usual inner product $\langle , \ . \ , \rangle$ and the canonical orthonormal basis $(e_1, e_2, ..., e_N)$. Further, we shall assume that R is reduced and crystallographic, that is a finite subset of $\mathbb{R}^N \backslash \{0\}$ which satisfies:
(i) R spanned $\mathbb{R}^N$.
(ii) R is invariant under r_α the reflection in the hyperplane orthogonal to any $\alpha \in R$.
(iii) $\alpha.\mathbb{R} \cap R = \{\pm\alpha\}$ for all $\alpha \in R$
(iii) for all $\alpha,\ \beta \in R$; $\langle \alpha, \breve{\beta}\rangle \in \mathbb{Z}$, $\breve{\beta} = \frac{2\beta}{\|\beta\|^2}$
We assume that the reader is familiar with the basics of root systems and their Weyl groups, see for examples Humphreys [9].

[1]

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1.a Heckman Opdam Jacobi polynomials.

Let R be a reduced root system with $\{\alpha_1, ..., \alpha_N\}$ be a basis of simple roots and R_+ be the set of positive roots determined by this basis. The fundamental weights $\{\beta_1, ..., \beta_N\}$ are given by $\langle\ \beta_j,\ \check{\alpha}_i\ \rangle = \delta_{i,j},\ \check{\alpha}_i = \dfrac{2\alpha_i}{\|\alpha_i\|^2}$. Let $Q = \bigoplus_{i=1}^{N} \mathbb{Z}\alpha_i,\ P = \bigoplus_{i=1}^{N} \mathbb{Z}\beta_i,\ Q^+ = \bigoplus_{i=1}^{N} \mathbb{N}\alpha_i$ and $P^+ = \bigoplus_{i=1}^{N} \mathbb{N}\beta_i$. We define a partial ordering on P by $\lambda \preceq \mu$ if $\mu - \lambda \in Q^+$

The group algebra $\mathbb{C}[P]$ of the free Abelian group P is the algebra generated by the formal exponentials $e^\lambda,\ \lambda \in P$ subject to the multiplication relation $e^\lambda e^\mu = e^{\lambda+\mu}$. The Weyl group W acts on $\mathbb{C}[P]$ by $we^\lambda = e^{w\lambda}$. The orbit-sums $m_\lambda = \sum_{\mu \in W.\lambda} e^\mu,\ \lambda \in P^+$ form a basis of $\mathbb{C}[P]^W$, the subalgebra of W-invariant elements of $\mathbb{C}[P]$. Here $W.\lambda$ denotes the W-orbit of λ.

Let $\mathbb{T} = \mathbb{R}^d / 2\pi\check{Q}$ where $\check{Q} = \bigoplus_{i=1}^{d} \mathbb{Z}\check{\alpha}_i$. The algebra $\mathbb{C}[P]$ can be realized explicitly as the algebra of polynomials on the torus $\mathbb{T}$ through the identification $e^\lambda(\dot{x}) = e^{i\langle\lambda,x\rangle}$ where $\dot{x} \in \mathbb{T}$ is the image of $x \in \mathbb{R}^d$. Let $k : R \to [0, +\infty[$ be a W-invariant function, called multiplicity function. We equip $C[P]^W$ with the inner product

$$(f, g)_k = \int_{\mathbb{T}} f(x)\overline{g(x)}\delta_k(x)dx$$

where

$$\delta_k = \prod_{\alpha \in R^+} \left| e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}} \right|^{2k_\alpha}$$

and dx is the Haar measure on $\mathbb{T}$.

The Heckman Opdam Jacobi polynomials are introduced by Heckman and Opdam [8] as the unique family of elements $P_\lambda \in \mathbb{C}[P]^W,\ \lambda \in P^+$ satisfying the following conditions:

(i) $P_\lambda = m_\lambda + \sum_{\mu \prec \lambda} a_{\lambda\mu} m_\mu$

(ii) $\langle P_\lambda, m_\mu \rangle = 0$ if $\mu \in P_+,\ \lambda \prec \mu$.

(Note that in [8], these polynomials are indexed by $-P_+$ instead of P_+). They form an orthogonal basis of $\mathbb{C}[P]^W$ and satisfy the second differential equation

$$\Big(\Delta + \sum_{\alpha \in R_+} k_\alpha \coth(\frac{1}{2}\langle x, \alpha\rangle)\partial_\alpha\Big) P_\lambda(x) = \langle \lambda, \lambda + \sum_{\alpha \in R_+} k_\alpha \alpha \rangle P_\lambda(x).$$

where Δ is the Laplace operator on $\mathbb{R}^N$.

The Cherednik operator $T_\xi,\ \xi \in \mathbb{R}^N$, associated with the root system R and the multiplicity k is defined by

$$T^k_\xi = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha,\ \xi \rangle \, \frac{1 - r_\alpha}{1 - e^\alpha} - \langle \rho_k,\ \xi \rangle,$$

where $\rho_k = \dfrac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha$. The hypergeometric function F_k is defined as the unique holomorphic W-invariant function on $\mathbb{C}^N \times (\mathbb{R}^N + iU)$ (U is a W-invariant neighborhood of 0) which satisfies the system of differential equations:

$$p(T_{e_1}, ...T_{e_N})F_k(\lambda, .) = p(\lambda)F_k(\lambda, .); \qquad F(\lambda, 0) = 1$$

for all $\lambda \in \mathbb{C}^N$ and all W-invariant polynomial p on $\mathbb{R}^N$. The Heckman opdam Jacobi polynomials are related to the hypergeometric function F_k by (see [7])

$$F_k(\lambda + \rho_k, x) = c(\lambda + \rho_k) P_\lambda(x); \qquad \lambda \in P^+, \ x \in \mathbb{R}^N, \tag{1.2}$$

where the function c is given on $\mathbb{R}^N$ by

$$c(\lambda) = \prod_{\alpha \in R^+} \frac{\Gamma(\langle \lambda, \check{\alpha} \rangle)\Gamma(\langle \rho, \check{\alpha} \rangle + k_\alpha)}{\Gamma(\langle \lambda, \check{\alpha} \rangle + k_\alpha)\Gamma(\langle \rho, \check{\alpha} \rangle)}. \tag{1.3}$$

1.b Jack polynomials

Let $k > 0$, the symmetric group S_N acts on the ring of polynomials $\mathbb{Q}(k)[x_1, ..., x_N]$ by

$$\tau p(x_1, ..., x_N) = p(x_{\tau(1)}, ..., x_{\tau(N)})$$

Let Λ_N the subspace of symmetric polynomials,

$$\Lambda_N = \{p \in \mathbb{C}[x_1, ..., x_N], \ \tau p = p, \ \forall \tau \in S_N\}.$$

We call partition all $\lambda = (\lambda_1, ...\lambda_N) \in \mathbb{N}^N$ such that $\lambda_1 \geq ... \geq \lambda_N$. The weight of a partition λ is the sum $|\lambda| = \lambda_1 + ... + \lambda_N$ and its length $\ell(\lambda) = \max\{j; \ \lambda_j \neq 0\}$. The set of all partitions are partially ordered by the dominance order:

$$\lambda \leq \mu \Leftrightarrow |\lambda| = |\mu| \quad \text{and} \quad \lambda_1 + \lambda_2 + ... + \lambda_i \leq \mu_1 + \mu_2 + ... + \mu_i$$

for all $i = 1, 2, ..., N$. The simplest basis of Λ_N is given by the monomial symmetric polynomials,

$$m_\lambda(x) = \sum_{\mu \in S_N \lambda} x_1^{\mu_1} ... x_N^{\mu_N}.$$

We define an inner product on Λ_N by

$$\langle f, g \rangle_k = \int_T f(z)\overline{g(z)} \prod_{i<j} |z_i - z_j|^{2k} dz$$

where $T = \{(z_1, ..., z_N) \in \mathbb{C}^N; |z_j| = 1, \ \forall \ 1 \leq j \leq N\}$ is the N-dimensional torus and dz is the haar measure on T. Jack symmetric polynomials j_λ indexed by a partitions λ can be defined as the unique polynomials such that

(i) $j_\lambda = m_\lambda + \sum_{\mu \prec \lambda} m_\mu$,

(ii) $\langle j_\lambda, m_\mu \rangle_k = 0$ if $\lambda \leq \mu$.

By a result of I. G. Macdonald ([13], p: 383) they form a family of orthogonal polynomials. Jack polynomials can be defined as eigenfunctions of certain Laplac-Beltrami type operator (coming in the theory of Calogero integrable systems and in random matrix theory),

$$L_k = \sum_{i=1}^{d} x_i^2 \frac{\partial^2}{\partial x_i^2} + 2k \sum_{i \neq j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}.$$

Jack polynomials j_λ are homogeneous of degree $|\lambda|$ and satisfy the compatibility relation

$$j_{(\lambda_1,...,\lambda_{N-1},0)}(x_1,...,x_{N-1},0) = j_{(\lambda_1,...,\lambda_{N-1})}(x_1,...,x_{N-1}). \tag{1.4}$$

The relationship between Heckman Opdam Jacobi polynomials and Jack polynomials can be illustrated as follows (see [2]): Let $\mathbb{V}$ be the hyperplane orthogonal to the vector $e = e_1 + ... + e_N$. In V we consider the root system of type A_{N-1},

$$R_A = \{\pm(e_i - e_j),\ 1 \le i < j \le N\}.$$

The fundamental weights are given by $\pi_N(\omega_i)$, $\omega_i = e_1 + ... + e_i$, where π_N denote the orthogonal projection along e onto V,

$$\pi_N(x) = x - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right) e = \left(x_1 - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right), ..., x_N - \frac{1}{N}\left(\sum_{i=1}^{N} x_i\right)\right)$$

and then $P_A^+ = \{\pi_N(\lambda), \lambda \text{ partition}\}$. The result is that:

$$j_\lambda(e^x) = P_{\pi_N(\lambda)}(x), \tag{1.5}$$

For all partition λ and all $x \in \mathbb{V}$ with $e^x = (e^{x_1}, ..., e^{x_N})$.

1.c Dunkl kernels and Dunkl-Bessel functions

The Dunkl operator D_ξ, $\xi \in \mathbb{R}^N$ associated with a root system R and a multiplicity function k is defined by

$$D_\xi = \partial_\xi + \sum_{\alpha \in R^+} k(\alpha)\langle \alpha, \xi\rangle \frac{1 - r_\alpha}{\langle \alpha, . \rangle}.$$

The Dunkl intertwining operator V_k is the unique isomorphism on the polynomials space $\mathbb{C}[\mathbb{R}^N]$ such that

$$V_k(1) = 1, \quad V_k(\mathcal{P}_n) = \mathcal{P}_n \quad \text{and} \quad D_\xi V_k = V_k \partial_\xi$$

where $\mathcal{P}_n$ is the subspace of homogeneous polynomials of degree $n \in \mathbb{N}$. For $r > 0$, V_k extends to a continuous linear operator on the Banach space

$$A_r = \{f = \sum_{n=0}^{\infty} f_n,\ f_n \in \mathcal{P}_n,\ \|f\|_{A_r} = \sum_{n=0}^{\infty} \sup_{|x| \le r} |f_n(x)| < \infty\}$$

by

$$V_k(f) = \sum_{n=0}^{\infty} V_k(f_n).$$

A remarkable result due to M. Rösler [14] says that for each $x \in \mathbb{R}^N$,

$$V_k(f)(x) = \int_{\mathbb{R}^d} f(\xi) d\mu_x(\xi)$$

where μ_x is a probability measure supported in $co(x)$ the convex hull of the orbit W.x.

The Dunkl kernel E_k is given by

$$E_k(x, y) = V_k(e^{\langle\ .\ ,\ y\ \rangle})(x) = \int_{\mathbb{R}^d} e^{\langle \xi, y\rangle} d\mu_x(\xi), \quad x \in \mathbb{R}^N,\ y \in \mathbb{C}^N$$

and having the following properties:

(i) For each $y \in \mathbb{C}^N$ the function $E_k(.,y)$ is the unique solution of eigenvalue problem:

$$D_\xi f(x) = \langle \xi, y\rangle f(x) \ \forall \quad \xi \in \mathbb{R}^N \text{ and } f(0) = 1.$$

(ii) E_k extends to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$ and for all $(x,y) \in \mathbb{C}^N \times \mathbb{C}^N$, $w \in W$ and $t \in \mathbb{C}$:

$$E_k(x,y) = E_k(y,x), \quad E_k(wx,wy) = E_k(x,y) \quad \text{and} \quad E_k(x,ty) = E_k(tx,y)$$

We define the Bessel function associated with R and k by,

$$J_k(x,y) = \frac{1}{|W|}\sum_{w\in W} E_k(x,wy).$$

The limit transition between hypergeometric functions F_k and Dunkl Bessel function is expressed by (see (2.21) of [14])

$$J_k(x,y) = \lim_{n\to+\infty} F_k(nx+\rho_k, \frac{y}{n}) \ . \tag{1.6}$$

According to these preliminaries we can now formulate the main result of this note.

2 Integral formula for J_k

The starting point is the following remarkable integral identity obtained by [10] which connecting jack polynomials of N variables to Jack polynomials of $N-1$ variables. For $\lambda = (\lambda_1, ..., \lambda_N) \in \mathbb{R}^N$ we use the notation $|\lambda| = \lambda_1 + ... + \lambda_N$.

Proposition 1 ([10])**.** *Suppose that the partition μ has less than N parts and $\lambda \in \mathbb{R}^N$ such that $\lambda_1 \geq ... \geq \lambda_N$. Then*

$$j_\mu(\lambda) = \frac{1}{U(\mu)V(\lambda)^{2k-1}}\int_{\lambda_2}^{\lambda_1} ... \int_{\lambda_N}^{\lambda_{N-1}} j_\mu(\nu)V(\nu)\Pi(\lambda,\nu)d\nu \tag{2.1}$$

where

$$U(\mu) = \prod_{j=1}^{N-1}\beta(\mu_j + (N-j)k, k), \quad V(\lambda) = \prod_{1\leq i<j\leq N}(\lambda_i - \lambda_j)$$

and

$$\Pi(\lambda,\nu) = \prod_{i\leq j}(\lambda_i - \nu_j)^{k-1}\prod_{i>j}(\nu_j - \lambda_i)^{k-1}.$$

We follow three simple steps that lead to our formula. All functions of N variables will be indexed by N and by $N-1$ if it considered as $N-1$ variables.
Step 1: For any partition $\mu = (\mu_1, ..., \mu_N)$ we set

$$\widetilde{\mu} = (\mu_1 - \mu_N, ..., \mu_{N-1} - \mu_N, 0) \quad \text{and} \quad \overline{\mu} = (\mu_1 - \mu_N, ..., \mu_{N-1} - \mu_N) \in \mathbb{R}^{N-1}.$$

By Homogeneity of Jack polynomials we have that

$$j_{\mu,N}(\lambda) = \left(\prod_{j=1}^{N}\lambda_j\right)^{\mu_N} j_{\widetilde{\mu},N}(\lambda)$$

and from (2.1) and (1.4) we may write

$$j_{\mu,N}(\lambda)=\frac{\left(\prod_{j=1}^{N}\lambda_j\right)^{\mu_N}}{U_N(\widetilde{\mu})V_N(\lambda)^{2k-1}}\int_{\lambda_2}^{\lambda_1}...\int_{\lambda_N}^{\lambda_{N-1}} j_{\overline{\mu},N-1}(\nu)V(\nu)\Pi(\lambda,\nu)d\nu.$$

Taking λ in $\mathbb{V}$ and making use a change of variables we get that

$$\begin{aligned} j_{\mu,N}(e^{\lambda}) &= \frac{1}{U_N(\widetilde{\mu})V_N(e^{\lambda})^{2k-1}}\int_{\lambda_2}^{\lambda_1}...\int_{\lambda_N}^{\lambda_{N-1}} e^{|\nu|}j_{\overline{\mu},N-1}(e^{\nu})V_{N-1}(e^{\nu})\Pi_N(e^{\lambda},e^{\nu})d\nu \\ &= \frac{1}{U_N(\widetilde{\mu})V_N(e^{\lambda})^{2k-1}}\int_{\lambda_2}^{\lambda_1}...\int_{\lambda_N}^{\lambda_{N-1}} e^{|\nu|(1+\frac{|\overline{\mu}|}{N-1})}j_{\overline{\mu},N-1}(e^{\pi_{N-1}(\nu)})V_{N-1}(e^{\nu})\Pi_N(e^{\lambda},e^{\nu})d\nu. \end{aligned}$$

In order by (1.2), (1.5) and (1.3) we have

$$F_N(\pi_N(\mu)+\rho_{k,N},\lambda)=c_N(\pi_N(\mu)+\rho_{k,N})j_{\mu,N}(e^{\lambda})=c_N(\mu+\rho_{k,N})j_{\mu,N}(e^{\lambda}),$$

where here

$$\rho_{k,N}=\frac{k}{2}\sum_{i=1}^{N}(N-2i+1)e_i=\left(\frac{k(N-1)}{2},...,\frac{k(N-2i+1)}{2},...,\frac{-k(N-1)}{2}\right)\in\mathbb{R}^N$$

and

$$c_N(\mu+\rho_{k,N})=\prod_{1\le i<j\le N}\frac{\Gamma(\mu_i-\mu_j)\Gamma(k(j-i+1))}{\Gamma(\mu_i-\mu_j+k)\Gamma(k(j-i))}.$$

Therefore,

$$\begin{aligned} F_N(\pi_N(\mu)+\rho_{k,N},\lambda) &= \frac{c_N(\mu+\rho_{k,N})}{c_{N-1}(\overline{\mu}+\rho_{k,N-1})U_N(\widetilde{\mu})V_N(e^{\lambda})^{2k-1}} \\ &\int_{\lambda_2}^{\lambda_1}...\int_{\lambda_N}^{\lambda_{N-1}} e^{|\nu|(1+\frac{|\overline{\mu}|}{N-1})}F_{N-1}(\pi_{N-1}(\overline{\mu})+\rho_{k,N-1},\pi_{N-1}(\nu))V_{N-1}(e^{\nu})\Pi_N(e^{\lambda},e^{\nu})d\nu. \end{aligned}$$

Step 2: Now we apply (1.6), by using the following when $n\to+\infty$

$$U_N(n\widetilde{\mu})\sim n^{-k(N-1)}\Gamma(k)^{N-1}\prod_{j=1}^{N-1}(\mu_j-\mu_N)^{-k},$$

$$V_N(e^{\frac{\lambda}{n}})\sim n^{-\frac{N(N-1)}{2}}V_N(\lambda),$$

$$c_N(n\mu+\rho_{k,N})\sim n^{\frac{-kN(N-1)}{2}}V_N(\mu)^{-k}\prod_{1\le i<j\le N}\frac{\Gamma(k(j-i+1))}{\Gamma(k(j-i))},$$

$$c_{N-1}(n\overline{\mu}+\rho_{k,N-1})\sim n^{\frac{-k(N-1)(N-2)}{2}}V_{N-1}(\overline{\mu})^{-k}\prod_{1\le i<j\le N-1}\frac{\Gamma(k(j-i+1))}{\Gamma(k(j-i))},$$

$$\frac{c_N(n\mu+\rho_{k,N})}{c_{N-1}(n\overline{\mu}+\rho_{k,N-1})}\sim n^{-k(N-1)}\frac{\Gamma(Nk)}{\Gamma(k)}\prod_{j=1}^{N-1}(\mu_j-\mu_N)^{-k},$$

$$\Pi_N(e^{\frac{\lambda}{n}},e^{\frac{\nu}{n}})\sim n^{-N(N-1)(k-1)}\Pi(\lambda,\nu).$$

Thus

$$J_{k,N}(\pi_N(\mu),\lambda) = \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N}\int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_N}^{\lambda_{N-1}} e^{|\overline{\mu}|\frac{|\nu|}{N-1}} J_{k,N-1}(\pi_{N-1}(\overline{\mu}),\pi_{N-1}(\nu))V(\nu)\Pi(\lambda,\nu)d\nu. \quad (2.2)$$

Step 3:

The formula (2.2) is valid only for a partition μ, to keep it for any $\mu\in\mathbb{R}^N$ we proceed as follows. Let $r\in(0,+\infty)$ and μ be a partition. We obtain after a change of variables

$$J_{k,N}(\pi_N(r\mu),\lambda) = J_{k,N}(\pi_N(\mu),r\lambda) = \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N}\int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_N}^{\lambda_{N-1}} e^{|\overline{r\mu}|\frac{|\nu|}{N-1}} J_{k,N-1}(\pi_{N-1}(\overline{r\mu}),\pi_{N-1}(\nu))V(\nu)\Pi(\lambda,\nu)d\nu.$$

Since the set $\{r\mu;\quad r\in(0,+\infty),\quad \mu \text{ partitions }\}$ is dense in the set

$$H=\{\mu\in\mathbb{R}^N,\quad 0\le\mu_N\le\ldots\le\mu_1\}$$

and $J_{k,N}$ is S_N-invariant continuous function then (2.2) can be extended to all $\mu\in H$. Now for $\mu\in\mathbb{R}^N$ we denote by μ^+ the unique element of $S_N.\mu$ so that $\mu_N^+\le\ldots\le\mu_1^+$. So we have $J_{k,N}(\pi_N(\mu),\lambda)=J_{k,N}(\pi_N(\mu^+),\lambda)=J_{k,N}(\pi_N(\widetilde{\mu^+}),\lambda)$ and since $\widetilde{\mu^+}\in H$ then

$$J_{k,N}(\pi_N(\mu),\lambda) = \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N}\int_{\lambda_2}^{\lambda_1}\cdots\int_{\lambda_N}^{\lambda_{N-1}} e^{|\overline{\mu^+}|\frac{|\nu|}{N-1}} J_{k,N-1}(\pi_{N-1}(\overline{\mu^+}),\pi_{N-1}(\nu))V(\nu)\Pi(\lambda,\nu)d\nu.$$

Now when restricted to the space $\mathbb{V}$ we state the following.

Theorem 1. *For all $\mu,\lambda\in\mathbb{V}$ we have*

$$J_{k,N}(\mu,\lambda) = \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N}\int_{\lambda_2^+}^{\lambda_1^+}\cdots\int_{\lambda_N^+}^{\lambda_{N-1}^+} e^{|\overline{\mu^+}|\frac{|\nu|}{N-1}} J_{k,N-1}(\pi_{N-1}(\overline{\mu^+}),\pi_{N-1}(\nu))V_{N-1}(\nu)\Pi_N(\lambda^+,\nu)d\nu. \quad (2.3)$$

In what follows, we shall restrict ourselves to the case $N=2,3,4$ where we give representations of $J_{k,N}$ as Laplace-type integrals,

$$J_{k,N}(\mu,\lambda)=\int_{\mathbb{R}^N} e^{\langle\mu,x\rangle}d\nu_\lambda(x).$$

where ν_λ is a probability measure supported in the convex hall of the orbit $S_N.\lambda$.

2.a Bessel function of type A_1

When $N=2$ we have that $\mathbb{V}=\mathbb{R}(e_1-e_2)$, $\mu^+=(|\mu_1|,-|\mu_1|)$, $\overline{\mu^+}=2|\mu_1|$ and $\lambda^+=(|\lambda_1|,-|\lambda_1|)$. It is obvious that $J_{k,1}=1$, so we get from (2.3)

$$\begin{aligned} J_{k,2}(\mu,\lambda) &= \frac{\Gamma(2k)}{(\Gamma(k))^2(2|\lambda_1|)^{2k-1}}\int_{-|\lambda_1|}^{|\lambda_1|} e^{2|\mu_1|\nu}(\lambda_1^2-\nu^2)^{k-1}d\nu \\ &= \frac{\Gamma(k+\frac12)}{\sqrt{\pi}\Gamma(k)}\int_{-1}^{1} e^{(2|\mu_1||\lambda_1|\nu}(1-\nu^2)^{k-1}d\nu \\ &= \mathcal{J}_{k-\frac12}(2\mu_1\lambda_1) \end{aligned}$$

where $\mathcal{J}_{k-\frac{1}{2}}$ is the modified Bessel function given by

$$\mathcal{J}_{k-\frac{1}{2}}(z) = \Gamma(k+\frac{1}{2})\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+k+\frac{1}{2})}(\frac{z}{2})^{2n}.$$

However, it is usual to identify $\mathbb{V} = \mathbb{R}\varepsilon$, $\varepsilon = \frac{e_1-e_2}{\sqrt{2}}$ with $\mathbb{R}$ and write

$$J_{k,2}(\mu,\lambda) = \mathcal{J}_{k-\frac{1}{2}}(\mu\lambda), \qquad \mu,\lambda\in\mathbb{R}.$$

2.b Bessel function of type A_2

Let $\mu = (\mu_1,\mu_2,\mu_3)$ and $\lambda = (\lambda_1,\lambda_2,\lambda_3)$ in the fundamental Weyl chamber

$$C = \{(u_1,u_2,u_3); \quad u_1 \geq u_2 \geq u_3, \quad u_1+u_2+u_3 = 0\}.$$

With $\overline{\mu} = (\mu_1-\mu_3,\mu_2-\mu_3,)$ and $\pi_2(\overline{\mu}) = (\frac{\mu_1-\mu_2}{2},\frac{\mu_2-\mu_1}{2})$ the formula (2.3) gives

$$\begin{aligned} J_{k,3}(\mu,\lambda) \quad = \quad & \frac{\Gamma(3k)}{V(\lambda)^{2k-1}\Gamma(k)^3}\int_{\lambda_2}^{\lambda_1}\int_{\lambda_3}^{\lambda_2} e^{\frac{(\mu_1+\mu_2-2\mu_3)(\nu_1+\nu_2)}{2}}\mathcal{J}_{k-\frac{1}{2}}(\frac{(\mu_1-\mu_2)(\nu_1-\nu_2)}{2})(\nu_1-\nu_2) \\ & \Big((\lambda_1-\nu_1)(\lambda_1-\nu_2)(\lambda_2-\nu_2)(\nu_1-\lambda_2)(\nu_1-\lambda_3)(\nu_2-\lambda_3)\Big)^{k-1}d\nu_1 d\nu_2. \end{aligned}$$

Using the change of variables: $x = \frac{\nu_1+\nu_2}{2}$, $z = \frac{\nu_1-\nu_2}{2}$ we have

$$\begin{aligned} J_{k,3}(\mu,\lambda) = \quad & \frac{4\Gamma(3k)}{V(\lambda)^{2k-1}\Gamma(k)^3}\int_{\mathbb{R}}\int_{\mathbb{R}} ze^{(\mu_1+\mu_2-2\mu_3)x}\mathcal{J}_{k-\frac{1}{2}}(\mu_1-\mu_2)z)\ \chi_{[\lambda_2,\lambda_1]}(x+z)\ \chi_{[\lambda_3,\lambda_2]}(x-z) \\ & \Big((\lambda_1-x)^2-z^2)((\lambda_3-x)^2-z^2)(z^2-(\lambda_2-x)^2)\Big)^{k-1}dxdz. \end{aligned} \tag{2.4}$$

Now recall that

$$\begin{aligned} \mathcal{J}_{k-\frac{1}{2}}((\mu_1-\mu_2)z) \quad &= \quad \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2}\int_{\mathbb{R}} e^{(\mu_1-\mu_2)zt}(1-t^2)^{k-1}\chi_{[-1,1]}(t)dt. \\ &= \quad \frac{\Gamma(2k)}{2^{2k-1}\Gamma(k)^2}\int_{\mathbb{R}} e^{(\mu_1-\mu_2)y}(1-\frac{y^2}{z^2})^{k-1}\chi_{[-1,1]}(\frac{y}{z})z^{-1}dy \end{aligned} \tag{2.5}$$

then inserting (2.5) in (2.4) with the use of Fubini's Theorem we can write

$$J_{k,3}(\mu,\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{(\mu_1+\mu_2-2\mu_3)x+(\mu_1-\mu_2)y}\Delta_k(\lambda,x,y)dxdy$$

where

$$\begin{aligned} &\Delta_k(\lambda,x,y) = \\ &\frac{4\Gamma(2k)\Gamma(3k)}{2^{2k-3}\Gamma(k)^5V(\lambda)^{2k-1}}\int_{\mathbb{R}}\left(\frac{z^2-y^2}{z^2}\right)^{k-1}\Big((\lambda_1-x)^2-z^2)((\lambda_3-x)^2-z^2)(z^2-(\lambda_2-x)^2)\Big)^{k-1} \\ &\qquad \chi_{[-1,1]}(\frac{y}{z})\chi_{[\lambda_1,\lambda_2]}(x+z)\chi_{[\lambda_3,\lambda_2]}(x-z)dz \end{aligned}$$

We note here that

$$\chi_{[-1,1]}(\frac{y}{z})\chi_{[\lambda_1,\lambda_2]}(x+z)\chi_{[\lambda_3,\lambda_2]}(x-z) = \chi_{\max(|y|,|x-\lambda_2|)\leq z\leq \min(x-\lambda_3,\lambda_1-x)}.$$

Thus we have

$$\Delta_k(\lambda,x,y) = \frac{4\Gamma(2k)\Gamma(3k)}{2^{2k-3}\Gamma(k)^5 V(\lambda)^{2k-1}} \int_{\max(|y|,|x-\lambda_2|)}^{\min(x-\lambda_3,\lambda_1-x)} \left(\frac{z^2-y^2}{z^2}\right)^{k-1} \Big((\lambda_1-x)^2-z^2)((\lambda_3-x)^2-z^2)(z^2-(\lambda_2-x)^2)\Big)^{k-1} dz$$

if

$$\max(|y|,|x-\lambda_2|) \leq \min(x-\lambda_3,\lambda_1-x)$$

and $\Delta_k(\lambda,x,y)=0$, otherwise. Making the change of variables

$$\nu_1 = x+y, \qquad \nu_2 = x-y$$

and put $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbb{V}$ with $\nu_3=-(\nu_1+\nu_2)$ we obtain

$$J_3^k(\mu,\lambda) = \frac{1}{2}\int_{\mathbb{R}^2} e^{\mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3}\Delta_{k,2}\left(\lambda,\frac{\nu_1+\nu_2}{2},\frac{\nu_1-\nu_2}{2}\right) d\nu_1 d\nu_2$$

But we can identify $\mathbb{R}^2$ with the space $\mathbb{V}$ via the basis (e_1-e_2,e_2-e_3), since for $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbb{V}$ we have $\nu=\nu_1(e_1-e_2)+\nu_2(e_1-e_3)$. Then we get

$$J_{k,3}(\mu,\lambda) = \int_{\mathbb{R}^2} e^{\langle\mu,\nu\rangle}\delta_{k,2}(\lambda,\nu)\, d\nu_1 d\nu_2. \tag{2.6}$$

with

$$\delta_{k,2}(\lambda,\nu) = \frac{1}{2}\Delta_{k,2}\left(\lambda,\frac{\nu_1+\nu_2}{2},\frac{\nu_1-\nu_2}{2}\right).$$

Now considering the orthonormal basis $(\varepsilon_1,\varepsilon_2)$ of $\mathbb{V}$,

$$\varepsilon_1 = \frac{1}{\sqrt{6}}(e_1+e_2-2e_3), \quad \varepsilon_2 = \frac{1}{\sqrt{2}}(e_1-e_2)$$

we can write

$$\mu = \frac{(\mu_1+\mu_2-2\mu_3)}{\sqrt{6}}\,\varepsilon_1 + \frac{\mu_1-\mu_2}{\sqrt{2}}\,\varepsilon_2$$

and for $x=x_1\varepsilon_1+x_2\varepsilon_2$

$$\langle\mu,x\rangle = \frac{(\mu_1+\mu_2-2\mu_3)}{\sqrt{6}}x_1 + \frac{\mu_1-\mu_2}{\sqrt{2}}x_2$$

Then using change of variables $x_1=\sqrt{6}\,x$ and $x_2=\sqrt{2}\,y$ in the formula (**??**) we obtain

$$J_3^k(\mu,\lambda) = \frac{1}{\sqrt{12}}\int_{\mathbb{R}^2} e^{\langle\mu,x\rangle}\Delta_k\left(\lambda,\frac{x_1}{\sqrt{6}},\frac{x_2}{\sqrt{2}}\right) dx_1 dx_2.$$

Proposition 2. *For all $\lambda=(\lambda_1,\lambda_2,\lambda_3)\in C$ the function $\delta_{k,2}(\lambda,.)$ is supported in the closed convex hull $co(\lambda)$ of the S_3-orbit of λ, described by: $\nu=(\nu_1,\nu_2,\nu_3)\in\mathbb{V}$ such that*

$$\lambda_3 \leq \min(\nu_1,\nu_2,\nu_3) \leq \max(\nu_1,\nu_2,\nu_3) \leq \lambda_1.$$

Proof. In view of (2.6) and (2.6) the support of $\delta_{k,2}(\lambda, .)$ is contain is the set

$$\left\{\nu \in \mathbb{V}; \quad \max\left(\frac{|\nu_1-\nu_2|}{2}, \left|\frac{\nu_1+\nu_2}{2}-\lambda_2\right|\right) \leq \min\left(\frac{\nu_1+\nu_2}{2}-\lambda_3, \lambda_1-\frac{\nu_1+\nu_2}{2}\right)\right\}$$

which by straightforward calculus reduced to the set

$$\{\nu \in \mathbb{V}; \quad \lambda_3 \leq \min(\nu_1,\nu_2,\nu_3) \leq \max(\nu_1,\nu_2,\nu_3) \leq \lambda_1\}.$$

However, we known that

$$\nu \in co(\lambda) \quad \Leftrightarrow \quad \lambda^+ - \nu^+ \in \bigoplus_{i=1}^{N} \mathbb{R}_+\alpha_i$$

and here

$$\lambda^+ - \nu^+ = \lambda - \nu^+ = (\lambda_1 - \nu_1^+)(e_1 - e_2) + (\nu_3^+ - \lambda_3)(e_2 - e_3)$$

Then

$$\nu \in co(\lambda) \quad \Leftrightarrow \quad \nu_1^+ \leq \lambda_1 \quad \text{and} \quad \nu_3^+ \geq \lambda_3,$$

which proves the proposition, since $\nu_1^+ = \max(\nu_1,\nu_2,\nu_3)$ and $\nu_3^+ = \min(\nu_1,\nu_2,\nu_3)$. □

2.c Bessel function of type A_3

Let $\mu, \lambda \in C$, the Weyl chamber. We have

$$\begin{aligned}
|\overline{\mu}| &= \mu_1+\mu_2+\mu_3-3\mu_4,\\
\pi_4(\overline{\mu}) &= (\mu_1+\frac{\mu_4}{3}, \mu_2+\frac{\mu_4}{3}, \mu_3+\frac{\mu_4}{3}),\\
\pi_4(\nu) &= (\frac{2\nu_1-\nu_2-\nu_3}{3}, \frac{2\nu_2-\nu_1-\nu_3}{3}, \frac{2\nu_3-\nu_1-\nu_2}{3}).
\end{aligned}$$

Taking (2.3) with the change of variables

$$\begin{aligned}
z_1 &= \frac{\nu_1+\nu_2+\nu_3}{3},\\
x_1 &= \frac{2\nu_1-\nu_2-\nu_3}{3},\\
x_2 &= \frac{2\nu_2-\nu_1-\nu_3}{3}
\end{aligned}$$

and put $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $x_1 + x_2 + x_3 = 0$, we have

$$J_{k,4}(\mu,\lambda) = \int_{\mathbb{R}^3} e^{(\mu_1+\mu_2+\mu_3-3\mu_4)z_1} J_{k,3}(\pi_3(\overline{\mu}), x) V_3(x)$$
$$\Pi_4(x_1+z_1, x_2+z_1, x_3+z_1, \lambda)\chi_{[\lambda_2,\lambda_1]}(x_1+z_1)\chi_{[\lambda_3,\lambda_2]}(x_2+z_1)\chi_{[\lambda_4,\lambda_3]}(x_3+z_1)dz_1dx_1dx_2.$$

By inserting (2.6)

$$\begin{aligned}
J_{k,4}(\mu,\lambda) \;=\; &\int_{\mathbb{R}^5} e^{\mu_1+\mu_2+\mu_3-3\mu_4)z_1+(\mu_1-\mu_3)z_2+(\mu_2-\mu_3)z_3}\delta_{k,2}((z_2,z_3,-(z_2+z_3)),x)V_3(x)\\
&\Pi_4(x_1+z_1, x_2+z_1, x_3+z_1, \lambda)\chi_{[\lambda_2,\lambda_2]}(x_1+z_1)\chi_{[\lambda_3,\lambda_2]}(x_2+z_1)\chi_{[\lambda_4,\lambda_3]}(x_3+z_1)\\
&\qquad dz_1dz_2dz_3dx_1dx_2.
\end{aligned}$$

Now with the change of variables

$$\begin{aligned} Z_1 &= z_1 + z_2, \\ Z_2 &= z_1 + z_3, \\ Z_3 &= z_1 - (z_2 + z_3) \end{aligned}$$

and with $Z = (Z_1, Z_2, Z_3, Z_4) \in \mathbb{R}^4$, such that $Z_1 + Z_2 + Z_3 + Z_4 = 0$ we have that

$$(\mu_1 + \mu_2 + \mu_3 - 3\mu_4)z_1 + (\mu_1 - \mu_3)z_2 + (\mu_2 - \mu_3)z_3 = \mu_1 Z_1 + \mu_2 Z_2 + \mu_3 Z_3 + \mu_4 Z_4 = \langle \mu, Z \rangle.$$

Therefore we can write

$$J_{k,3}(\mu, \lambda) = \int_{\mathbb{R}^3} e^{\langle \mu, Z \rangle} \delta_{k,3}(Z, \lambda) dZ_1 dZ_2 dZ_3,$$

where

$$\begin{aligned} &\delta_{k,3}(Z,\lambda) = \\ &\int_{\mathbb{R}^2} \Pi_4(x_1 + \frac{1}{3}(Z_1 + Z_2 + Z_3), x_2 + \frac{1}{3}(Z_1 + Z_2 + Z_3), x_3 + \frac{1}{3}(Z_1 + Z_2 + Z_3), \lambda) \\ &\delta_{k,2}(\frac{1}{3}(2Z_1 - Z_2 - Z_3), \frac{1}{3}(2Z_2 - Z_1 - Z_3), \frac{1}{3}(2Z_3 - Z_1 - Z_2), x) \chi_{[\lambda_2,\lambda_1]}(x_1 + \frac{1}{3}(Z_1 + Z_2 + Z_3)) \\ &\chi_{[\lambda_3,\lambda_2]}(x_2 + \frac{1}{3}(Z_1 + Z_2 + Z_3)) \chi_{[\lambda_4,\lambda_3]}(x_3 + \frac{1}{3}(Z_1 + Z_2 + Z_3)) dx_1 dx_2. \end{aligned} \tag{2.7}$$

Let us now describe the support of $\delta_{k,3}$. In fact, $\delta_{k,3}(Z,\lambda) \neq 0$ if the variables x and Z of the integrant (2.7) satisfy:

$$\begin{aligned} &(1) \qquad \lambda_2 \leq x_1 + \frac{1}{3}(Z_1 + Z_2 + Z_3) \leq \lambda_1, \\ &(2) \qquad \lambda_3 \leq x_2 + \frac{1}{3}(Z_1 + Z_2 + Z_3) \leq \lambda_2, \\ &(3) \qquad \lambda_4 \leq x_3 + \frac{1}{3}(Z_1 + Z_2 + Z_3) \leq \lambda_3, \\ &(4) \qquad x_3 \leq \frac{1}{3}(2Z_1 - Z_2 - Z_3) \leq x_1, \\ &(5) \qquad x_3 \leq \frac{1}{3}(2Z_2 - Z_1 - Z_3) \leq x_1, \\ &(6) \qquad x_3 \leq \frac{1}{3}(2Z_3 - Z_1 - Z_2) \leq x_1. \end{aligned}$$

It Follows that

$$\begin{array}{rcl}
(1)+(4) & \Rightarrow & Z_1 \le \lambda_1, \\
(1)+(5) & \Rightarrow & Z_2 \le \lambda_1, \\
(1)+(6) & \Rightarrow & Z_3 \le \lambda_1, \\
(1)+(2)+(3) & \Rightarrow & Z_4 \le \lambda_1, \\
(1)+(2)-(6) & \Rightarrow & Z_1+Z_2 \le \lambda_1+\lambda_2, \\
(1)+(2)-(5) & \Rightarrow & Z_1+Z_3 \le \lambda_1+\lambda_2, \\
(1)+(2)-(4) & \Rightarrow & Z_2+Z_3 \le \lambda_1+\lambda_2, \\
(2)+(3)-(4) & \Rightarrow & Z_1+Z_4 \le \lambda_1+\lambda_2, \\
(2)+(3)-(5) & \Rightarrow & Z_2+Z_4 \le \lambda_1+\lambda_2 \\
(2)+(3)-(6) & \Rightarrow & Z_3+Z_4 \le \lambda_1+\lambda_2, \\
(1)+(2)+(3) & \Rightarrow & Z_1+Z_2+Z_3 \le \lambda_1+\lambda_2+\lambda_3, \\
(3)+(4) & \Rightarrow & Z_2+Z_3+Z_4 \le \lambda_1+\lambda_2+\lambda_3, \\
(3)+(5) & \Rightarrow & Z_1+Z_3+Z_4 \le \lambda_1+\lambda_2+\lambda_3, \\
(3)+(6) & \Rightarrow & Z_1+Z_2+Z_4 \le \lambda_1+\lambda_2+\lambda_3.
\end{array}$$

These inequalities can be expressed in terms of $Z^+ = (Z_1^+, Z_2^+, Z_3^+, Z_4^+)$ as

$$\begin{aligned}
&Z_1^+ = \max(Z_1, Z_2, Z_3, Z_4) \le \lambda_1 \\
&Z_1^+ + Z_2^+ = \max(Z_1+Z_2, Z_1+Z_3, Z_1+Z_4, Z_2+Z_3, Z_2+Z_4, Z_3+Z_4) \le \lambda_1+\lambda_2 \\
&Z_1^+ + Z_2^+ + Z_3^+ = \max(Z_1+Z_2+Z_3, Z_2+Z_3+Z_4, Z_1+Z_2+Z_4, Z_1+Z_3+Z_4) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \le \lambda_1+\lambda_2+\lambda_3
\end{aligned}$$

which imply that $Z^+ \preceq \lambda$ and therefore $Z \in co(\lambda)$.

2.d Case for arbitrary N

After having idea about the case $N = 2, 3$ it is not hard to see that the formula (1.1) can be found using recurrence. In fact, let $\mu, \lambda \in C$ the Weyl chamber. Put for $\nu \in \mathbb{R}^{N-1}$

$$\Omega(\lambda, \nu) = \prod_{i=1}^{N-1} \chi_{[\lambda_{i+1}, \lambda_i]}(\nu).$$

With the change of variables

$$\begin{aligned}
z_1 &= \frac{|\nu|}{N-1} = \frac{\nu_1 + ... + \nu_{N-1}}{N-1} \\
x_i &= \nu_i - \frac{|\nu|}{N-1}; \qquad 1 \le i \le N-2
\end{aligned}$$

the formula (2.3) becomes,

$$\begin{aligned}
J_{k,N}(\mu, \lambda) \quad = \quad & \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N} \int_{\mathbb{R}^{N-1}} e^{|\overline{\mu}|z_1} J_{k,N-1}(\pi_{N-1}(\overline{\mu}), x) V_{N-1}(x) \\
& \Pi_N(\lambda, (x_1+z_1, ..., x_{N-1}+z_1))\Omega(\lambda, (x_1+z_1, ..., x_{N-1}+z_1)) dz_1 dx_1 ... dx_{N-2}
\end{aligned}$$

where we put $x = (x_1, ..., x_{N-1})$ with $x_{N-1} = -(x_1 + ... + x_{N-2})$. The recurrence hypothesis says that

$$\begin{aligned} J_{k,N-1}(\pi_{N-1}(\overline{\mu}), x) &= \int_{\mathbb{V}_{N-1}} e^{\langle \pi_{N-1}(\overline{\mu}), z\rangle} \delta_{k,N-1}(x,z)dz. \\ &= \int_{\mathbb{R}^{N-2}} e^{\sum_{i=1}^{N-1}\left(\overline{\mu}_i - \frac{|\overline{\mu}|}{N-1}\right)z_{i+1}} \delta_{k,N-1}(x,z)dz_2...dz_{N-2}. \end{aligned}$$

where $z = (z_2, ..., z_N)$ with $z_N = -(z_2 + ... + z_{N-1})$ and $\delta_{k,N-1}(., x)$ is supported in the convex hull of $S_{N-1}.x$ in $\mathbb{R}^{N-1}$. Hence we get

$$\begin{aligned} J_{k,N}(\mu, \lambda) &= \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-2}} e^{|\overline{\mu}|z_1 + \sum_{i=1}^{N-1}\left(\overline{\mu}_i - \frac{|\overline{\mu}|}{N-1}\right)z_{i+1}} \delta_{k,N-1}(z,x) \\ & \quad V_{N-1}(x)\Pi_N(\lambda, (x_1+z_1, ..., x_{N-1}+z_1))\Omega(\lambda, (x_1+z_1, ..., x_{N-1}+z_1)) \\ & \quad dz_1dz_2...dz_{N-1}dx_1...dx_{N-2}. \end{aligned}$$

Now observing that

$$\begin{aligned} |\overline{\mu}|z_1 + \sum_{i=1}^{N-1}\left(\overline{\mu}_i - \frac{|\overline{\mu}|}{N-1}\right)z_{i+1} &= \left(\sum_{i=1}^{N-1}\mu_i - (N-1)\mu_N\right)z_1 + \sum_{i=1}^{N-1}\mu_i z_{i+1} \\ &= \sum_{i=1}^{N-1}\mu_i(z_1 + z_{i+1}) - (N-1)\mu_{N-1}z_1. \end{aligned}$$

Then making the change of variables

$$Z_i = z_1 + z_{i+1}, \qquad 1 \le i \le N-1$$

and put $Z = (Z_1, ..., Z_N)$ with $Z_N = -(Z_1 + ... + Z_{N-1}$, we have

$$\begin{aligned} J_{k,N}(\mu, \lambda) &= \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N} \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-2}} e^{\sum_{i=1}^{N}\mu_i Z_i} \delta_{k,N-1}(\phi(Z), x)V_{N-1}(x)\Pi_N(\lambda, \theta(Z,x)) \\ & \quad \Omega_N(\lambda, \theta(Z,x))dZ_1dZ_2...dZ_{N-1}dx_1...dx_{N-2} \\ &= \int_{\mathbb{R}^{N-1}} e^{\sum_{i=1}^{N}\mu_i Z_i} \delta_{k,N}(\lambda, Z)dZ_1...dZ_{N-1} \\ &= \int_{\mathbb{V}_N} e^{\langle \mu, Z\rangle} \delta_{k,N}(\lambda, Z)dZ, \end{aligned}$$

with

$$\begin{aligned} \phi(Z) &= \left(Z_1 - \frac{\sum_{i=1}^{N-1} Z_i}{N-1}, ..., Z_{N_{-1}} - \frac{\sum_{i=1}^{N-1} Z_i}{N-1}\right) \\ \theta(Z,x) &= \left(x_1 + \frac{\sum_{i=1}^{N-1} Z_i}{N-1}, ..., x_{N-1} + \frac{\sum_{i=1}^{N-1} Z_i}{N-1}\right) \\ \delta_{k,N}(\lambda, Z) &= \frac{\Gamma(Nk)}{V_N(\lambda)^{2k-1}\Gamma(k)^N} \int_{\mathbb{R}^{N-2}} \delta_{k,N-1}(\phi(Z), x) \\ & \quad V_{N-1}(x)\Pi_N(\lambda, \theta(Z,x))\Omega_N(\lambda, \theta(Z,x))dx_1...dx_{N-2}. \end{aligned} \tag{2.8}$$

Now we write sufficient conditions for which the integrant (2.8) does not vanish

$$(\Lambda_i) \qquad \lambda_{i+1} \leq x_i + \frac{\sum_{i=1}^{N-1} Z_i}{N-1} \leq \lambda_i$$

$$(\Lambda_I) \qquad \sum_{i\in I} Z_i - \frac{|I|}{N-1}\sum_{i=1}^{N-1} Z_i \leq \sum_{i=1}^{|I|} x_i$$

for all $I \subset \{1, 2, ..., N-1\}$ of cardinally $|I|$. It follows that

$$\sum_{i=1}^{|I|} \Lambda_i + \Lambda_I \quad \Rightarrow \quad \sum_{i\in I} Z_i \leq \sum_{i=1}^{|I|} \lambda_i$$

which proves that $Z^+ \leq \lambda$ and then $Z \in co(\lambda)$.

3 Partially product formula for J_k

We will first establish a product formula for J_k provided that a conjecture of Stanley on the multiplication of Jack polynomials is true. The conjecture says that for all partitions μ and λ

$$j_\mu j_\lambda = \sum_{\nu\leq\mu+\lambda} g^\nu_{\mu,\lambda} j_\nu$$

where $g^\nu_{\mu,\lambda}$ (the Littlewood-Richardson coefficients) is a polynomial in k with nonnegative integer coefficients. In particular, $g^\nu_{\mu,\lambda} \geq 0$, what is the interesting facts in our setting. Hence we have for all μ, λ partitions,

$$F(\pi(\mu) + \rho_k, .)F(\pi(\lambda) + \rho_k, .) = \sum_{\nu\leq\mu+\lambda} f^\nu_{\mu,\lambda} F(\pi(\nu) + \rho_k, .)$$

with $f^\nu_{\mu,\lambda} \geq 0$ and

$$\sum_\nu f^\nu_{\mu,\lambda} = 1$$

But if $\nu \leq \mu + \lambda$ as partitions then we also have $\pi(\nu) \preceq \pi(\mu) + \pi(\lambda)$ in the dominance ordering ([2], Lemma 3.1). This allows us to write for all $\mu, \lambda \in P^+$

$$F(\mu + \rho_k, .)F(\lambda + \rho_k, .) = \sum_{\nu\in P^+;\, \nu\preceq\mu+\lambda} f^\nu_{\mu,\lambda} F(\nu + \rho_k, .).$$

To arrive at product formula for J_k we follow the technic used by M. Rösler in [14]. We first write

$$F(n\mu + \rho_k, \frac{z}{n})F(n\lambda + \rho_k, \frac{z}{n}) = \int_{\mathbb{R}^N} F(nx + \rho_k, \frac{z}{n}) d\gamma^n_{\mu,\lambda}(x), \qquad z \in \mathbb{V}.$$

where

$$d\gamma^n_{\mu,\lambda} = \sum_{\nu\in P^+;\, \nu\preceq\mu+\lambda} f^\nu_{\mu,\lambda}\, \delta_{\frac{\nu}{n}}.$$

According to ([14], Lemma 3.2) the probability measure $\gamma^n_{\mu,\lambda}$ is supported in the convex hull $co(\mu+\lambda)$. So, from Prohorov's theorem (see [3]) there exists a probability measure $\gamma_{\mu,\lambda}$ supported in

$co(\mu+\lambda)$ and a subsequence $(\gamma_{\mu,\lambda}^{n_j})_j$ which converges weakly to $\gamma_{\mu,\lambda}$. Then by using (1.6) it follows that

$$J_k(\mu,z)J_k(\lambda,z)=\int_{\mathbb{V}} J_k(\xi,z)d\gamma_{\mu,\lambda}(\xi)$$

for all $z\in\mathbb{V}$ and $\mu,\lambda\in P^+$.
Now let $r,s\in\mathbb{Q}^+$ with $r=\frac{a}{b}$ and $s=\frac{c}{b}$, $a,b,c\in\mathbb{N}$, $b\neq 0$. We write

$$\begin{aligned}
J_k(r\mu,z)J_k(s\lambda,z) &= J_k(a\mu,\frac{z}{b})J_k(c\lambda,\frac{z}{b})\\
&= \int_{\mathbb{V}} J_k(\xi,\frac{z}{b})d\gamma_{a\mu,c\lambda}(\xi); \quad z\in\mathbb{R}^d\\
&= \int_{\mathbb{V}} J_k(\frac{\xi}{b},z)d\gamma_{a\mu,c\lambda}(\xi); \quad z\in\mathbb{R}^d
\end{aligned}$$

Defining $\gamma_{r\mu,s\lambda}$ as the image measure of $\gamma_{a\mu,c\lambda}$ under the dilation $\xi\to\frac{\xi}{b}$. We get

$$J_k(r\mu,z)J_k(s\lambda,z)=\int_{\mathbb{V}} J_k(\xi,z)d\gamma_{r\mu,s\lambda}(\xi).$$

Now we apply the density argument, since $\mathbb{Q}^+.P^+\times\mathbb{Q}^+.P^+$ is dense in $C\times C$, where C is the Weyl chamber. Then Prohorov's theorem yields

$$J_k(\mu,z)J_k(\lambda,z)=\int_{\mathbb{V}} J_k(\xi,z)d\gamma_{\mu,\lambda}(\xi); \quad z\in\mathbb{R}^d$$

for all $\mu,\lambda\in C$ with $supp(\gamma_{\mu,\lambda})\subset co(\mu+\lambda)$. This finish our approach for the product formula.

An important special case of the Stanley conjecture called Peiri formula is where the partition $\lambda=(n)$, $n\in\mathbb{N}$. Since this formula has already been proved (see [16]) then we can state the following partial result

Theorem 2. *For all $\mu\in C$ and all $t\geq 0$ there exists a probability measure $\gamma_{\mu,t}$ such that*

$$J_k(\mu,z)J_k(t\beta_1,z)=\int_{\mathbb{V}} J_k(\xi,z)d\gamma_{\mu,t}(\xi); \quad z\in\mathbb{R}^d$$

where $\beta_1=\pi(e_1)$. The measure $\gamma_{\mu,t}$ is supported in $co(\mu+t\beta_1)$.

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